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A kernel-based classifier on a Riemannian manifold

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Summary: Let X be a random variable taking values in a compact Riemannian manifold without boundary, and let Y be a discrete random variable valued in $\{0; 1\}$ which represents a classification label. We introduce a kernel rule for classification on the manifold based on n independent copies of (X, Y). Under mild assumptions on the bandwidth sequence, it is shown that this kernel rule is consistent in the sense that its probability of error converges to the Bayes risk with probability one.

1 Introduction

In many experiments, the intrinsic structure of the collected data is no longer Euclidean; instead, the observations are points of a given Riemannian manifold. For instance the sphere is the sample space in axial and directional statistics (Fisher et al., 1993; Mardia and Jupp, 2000; Watson, 1983). Three-dimensional rotations or rigid transformations are considered in medical image analysis and high level computer vision (see e.g. Pennec, 2006 and the references therein). Other examples of manifolds encountered in statistical applications include the Stiefel manifold (i.e., the space of k-frames in $\mathbb{R}^m$) and the Grassman manifold $G_{k,m-k}$ (i.e., the space of k-dimensional hyperplanes in $\mathbb{R}^m$) thoroughly studied by Chikuse (2003), or the manifold of shapes characterized by a corpus of landmarks (Dryden and Mardia, 1998; Kendall et al., 1999; Le and Kendall, 1993; Mardia and Patrangenaru, 2005; Small, 1996).

Stimulated by multiple applications, there is presently a growing literature on statistical inference on manifolds, and on the related topic of manifold learning, i.e., the problem of inferring a low-dimensional, possibly nonlinear, structure underlying the data. The two subjects are complementary one to each other, the fact of the matter being that fitting an appropriate submanifold to the data allows for efficient dimension reduction. On the other hand, using a nonlinear representation of the data requires the adaptation of the existing statistical procedures. In this display, when a manifold is given as a sample space, several results have been obtained, including the estimation of location parameters (Bhattacharya and Patrangenaru, 2003, 2005), density and regression estimation (Hendriks, 1990; Hendriks et al., 1993; Lee and Ruymgaart, 1996; Kim, 1998; Pelletier, 2005,

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2006), and goodness-of-fit tests (see Jupp (2005) for recent results and further references). Recently, density estimation on an unknown submanifold of $\mathbb{R}^m$ has been considered in Hein (2006); see also the results in Hein et al. (2007) and Giné and Koltchinskii (2006). However, few is known about classification on a given manifold. Indeed, parametric methods are considered in El Khattabi and Streit (1996) and Hayakawa (1997) in the context of directional statistics, i.e. on the sphere, and to the best of our knowledge, no results are available for the nonparametric classification of observations on a general manifold. The aim of the present paper is to generalize the Euclidean kernel rule for the classification of observations belonging to a given compact Riemannian manifold without boundary.

Classification consists in predicting the unknown label $Y \in \{0, 1\}$ of an observation $X \in \mathcal{X}$. It is also called *discrimination* or *supervised classification*, this latter terminology being frequently used in the machine learning community, and we will simply use the term classification for short. The observation X as well as its label Y are assumed to be random so that the frequency of outcome of particular pairs is described by the distribution of (X, Y). In practice, the classification procedure is performed by a *classifier* or *classification rule*, which in mathematical terms is defined as a function $g : \mathcal{X} \to \{0, 1\}$. The performance of a given classifier g may be quantified by its probability of error $L(g)$ defined by

$$L(g) = \mathbf{P}(g(X) \neq Y),$$

an error occurring whenever $g(X) \neq Y$. It is well known (see e.g., Devroye et al., 1996 for a recent exposition) that the minimum of $L(g)$ over all possible classifiers g is achieved by the Bayes rule given by

$$g^\star(x) = \begin{cases} 0 & \text{if } \mathbf{P}(Y = 0|X = x) \geq \mathbf{P}(Y = 1|X = x) \\ 1 & \text{otherwise.} \end{cases} \tag{1.1}$$

In this sense, the Bayes rule is the optimal decision. However, it depends on the unknown distribution of the pair (X, Y), and for this reason, the Bayes classifier cannot be constructed in practice. Therefore, we shall consider an *empirical classifier* g_n based on n independent copies $(X_1, Y_1), \ldots, (X_n, Y_n)$ of (X, Y). Following Devroye et al. (1996), the classifier g_n will be called *strongly consistent* if its probability of error

$$L(g_n) = \mathbf{P}(g_n(X) \neq Y|(X_1, Y_1), \ldots, (X_n, Y_n))$$

is such that

$$\lim_{n\to\infty} L(g_n) = L(g^\star) \quad \text{with probability one.}$$

In the present paper, we focus on the kernel classification rule, which is derived from kernel density estimate, pioneered in Akaike (1954), Parzen (1962) and Rosenblatt (1956). More precisely in a Euclidean space, the kernel rule consists in labeling by 0 a point x if $\sum_{i=1}^n \mathbf{1}_{\{Y_i=0\}} K((x - X_i)/h_n) \geq \sum_{i=1}^n \mathbf{1}_{\{Y_i=1\}} K((x - X_i)/h_n)$, and by 1 otherwise, where the kernel K is a nonnegative function decreasing with the distance to the origin, and where h_n is a sequence of smoothing parameters. Using the kernel introduced in Pelletier (2005, 2006), we generalize herein the kernel classification rule to the case of a closed Riemannian manifold and we prove its strong consistency.

The paper is organized as follows. Section 2 introduces the kernel on the manifold defined in Pelletier (2005) as well as some notation. In Section 3, we define the kernel classification rule and prove its strong consistency. For clarity, the proof of our main result, which relies on several auxiliary results, is exposed in Section 4.

2 Notation and definitions

Let (M, g) be a compact Riemannian manifold without boundary of dimension d. Let us first briefly introduce the main notions from Riemannian geometry used throughout the paper. For materials on differential geometry, we refer to Chavel (1993) and Kobayashi and Nomizu (1969). First of all, we shall denote by d_g the *Riemannian geodesic distance*, and by v_g the *Riemannian volume measure* on M. The Riemannian metric tensor g defines a scalar product on each tangent space T_pM to M at p. Given a chart (U, φ) on M with domain U and local coordinates $x_1, \ldots, x_d$ on $\varphi(U) \subset \mathbb{R}^d$, the local representation of the Riemannian metric on U is given by the $d \times d$ matrix the elements of which are defined by $g_{ij} = g(\partial_i, \partial_j)$, where ∂_i denotes the i^{th} coordinate vector field on U. Locally, the *volume element* is expressed as $\sqrt{|g(x)|}$, where $|g(x)|$ denotes the determinant of the metric in local coordinates, i.e., given an integrable function f on M whose support is included in the domain U of the chart (U, φ), we have $\int_U f(p) v_g(\mathrm{d}p) = \int_{\varphi(U)} f\left(\varphi^{-1}(x)\right)\sqrt{|g(x)|}\lambda(\mathrm{d}x)$, where λ denotes the Lebesgue measure on $\mathbb{R}^d$.

Given a point $p \in M$, the *exponential map* at p, denoted by $\exp_p$ and defined on a suitable neighborhood of 0_p in T_pM, maps a tangent vector X_p to M at p to the point of M located at the distance $\|X_p\|$ on the unique geodesic starting at p with initial velocity vector X_p. When the Riemannian manifold is complete, which is the case herein, $\exp_p$ is indeed defined on all of T_pM. Every point p in M has a neighborhood U which is the diffeomorphic image of a star-shaped neighborhood of 0_p in T_pM. So using the canonical identification of T_pM with $\mathbb{R}^d$, the inverse $\exp_p^{-1}$ of $\exp_p$ gives rise to local coordinates called *geodesic normal coordinates*. In these local coordinates, geodesics through p are represented as straight lines, thus $\exp_p$ locally rectifies geodesics. At last, at each p in M, there exists a maximal number $\mathrm{inj}_g(p) > 0$, called the *injectivity radius at* p, such that the restriction of the exponential map at p to the ball in T_pM centered at 0_p and of radius $\mathrm{inj}_g(p)$ is a diffeomorphism onto its image. The infimum over p of all the injectivity radii at p is called the *injectivity radius* of M, and is denoted by $\mathrm{inj}_g(M)$.

We are now in a position to define a kernel K_h on a Riemannian manifold (M, g) with bandwidth parameter h, as in Pelletier (2005). First of all, let $K : \mathbb{R}_+ \to \mathbb{R}$ be a positive and continuous map such that:

(i) $\int_{\mathbb{R}^d} K(\|u\|)\lambda(\mathrm{d}u) = 1$,

(ii) $\operatorname{supp} K = [0; 1]$,

where λ denotes the Lebesgue measure on $\mathbb{R}^d$.

Now for p and q two points of M, let $\theta_p(q)$ be the volume density function on M roughly defined by Besse (1978, p. 154):

$$\theta_p : q \mapsto \theta_p(q) = \frac{\mu_{\exp_p^* g}}{\mu_{g_p}}(\exp_p^{-1}(q)),$$

i.e., the quotient of the canonical measure of the Riemannian metric $\exp_p^* g$ on $T_p(M)$ (pullback of g by the map $\exp_p$) by the Lebesgue measure of the Euclidean structure g_p on $T_p(M)$. Note that this definition makes sense for q in a neighborhood of p, yet the volume density function may be defined globally by recursing to Jacobi fields (Willmore, 1993, p. 219). More importantly, in terms of geodesic normal coordinates at p, $\theta_p(q)$ equals the square root of the determinant of the metric g expressed in these coordinates at $\exp_p^{-1}(q)$, i.e., the volume element expressed in geodesic normal coordinates. Additionally, for p and q in a normal neighborhood U of M, we have $\theta_p(q) = \theta_q(p)$ (Willmore, 1993, p. 221).

Then we define a kernel $K_h(p, \cdot) : M \to \mathbb{R}_+$ on M by:

$$K_h(p, q) = \frac{1}{\theta_p(q)} \frac{1}{h^d} K\left(\frac{d_g(q, p)}{h}\right), \tag{2.1}$$

for all $q \in M$. In (2.1), h is the *bandwidth* or *smoothing parameter* and we assume that it satisfies the condition

$$h \leq h_0 < \mathrm{inj}_g(M), \tag{2.2}$$

for some fixed h_0, where $\mathrm{inj}_g(M)$ is the injectivity radius of M [strictly positive since M is compact].

The kernel (2.1) has some interesting properties proved in Pelletier (2005) that we briefly summarize below. First of all, this kernel is a probability density on M with respect to the Riemannian volume measure. Second, if the function K is such that $\int_{\mathbb{R}^d} uK(\|u\|)\lambda(du) = 0$, then the kernel is centered on p in the sense that, if a random variable X valued in M has density $K_h(p, \cdot)$ with respect to v_g, then p is the intrinsic mean of X, provided h is small enough. Additionally, when M is $\mathbb{R}^d$, we have $\theta_p(q) = 1$ for all p, q, and so K_h reduces to a standard isotropic kernel on $\mathbb{R}^d$ supported by the closed unit Euclidean ball.

In all of the following, we shall assume that the function K is such that

$$\inf_{0 \leq x \leq \frac{1}{2}} K(x) > 0,$$

which implies that the kernel $K_h(p, \cdot)$ takes strictly positive values on the geodesic ball $B_M(p, \frac{h}{2})$ centered at p and of radius $h/2$. This assumption is needed in the proofs of Lemma 4.3 and Lemma 4.5 and is related to the notion of *regular* kernels on $\mathbb{R}^d$ (see e.g., Devroye et al., 1996, Definition 10.1). In this assumption, the scalar $\frac{1}{2}$ is arbitrary. It could be replaced by any real number in the open interval (0; 1), and the particular value of $\frac{1}{2}$ is selected for the sake of simplicity only.

3 Kernel classification rule

In this section, we define a kernel classification rule using the kernel (2.1) and establish its consistency. To this aim, let $(X_1, Y_1), \ldots (X_n, Y_n)$ be n independent copies of a pair of random variables (X, Y) valued in $M \times \{0; 1\}$. Then we define the *kernel classification rule* $g_n^0 : M \to \{0; 1\}$ by:

$$g_n^0(p) = \begin{cases} 0, & \text{if } \sum_{i=1}^n \mathbf{1}_{\{Y_i=0\}} K_{h_n}(p, X_i) \geq \sum_{i=1}^n \mathbf{1}_{\{Y_i=1\}} K_{h_n}(p, X_i), \\ 1, & \text{otherwise,} \end{cases} \tag{3.1}$$

for all $p \in M$, and where K_{h_n} is a kernel on M of the form given by (2.1) with bandwidth sequence h_n.

As in the Introduction, $L(g^\star)$ will denote the probability of error of the Bayes rule $g^\star$ defined by (1.1), and the classification error probability of the kernel rule will be denoted by $L\left(g_n^0\right)$, i.e.,

$$L\left(g_n^0\right) = \mathbf{P}(g_n^0(X) \neq Y | (X_1, Y_1), \ldots, (X_n, Y_n)).$$

We are now in a position to state our main result.

Theorem 3.1 *Suppose that $h_n \to 0$ and $nh_n^d \to \infty$. Then*

$$\lim_{n\to\infty} L\left(g_n^0\right) = L(g^\star)$$

with probability one.

Theorem 3.1 states that the kernel classification rule (3.1) is strongly consistent. As exposed in the Introduction, the application field of this type of result is vast, in particular when the data to be classified does not have an Euclidean structure, and thus preventing the use of standard classification method. It is the case for instance when studying automatic labelling of shapes which, following Kendall et al. (1999), can be understood as points on a Riemannian manifold. However, the practical implementation of the kernel rule (3.1) requires knowledge of the geometry of the manifold. Depending on the application, if the geometry is unknown, an extra work would be needed to derive the geometric quantities, or a numerical approximation to them. Those aspects exceed the scope of the present paper and are left for future research.

4 Proofs

The proof of Theorem 3.1 is given in Section 4.3 and relies on several auxiliary results. One first Lemma on the metric entropy of the manifold is proved in Section 4.1. Auxiliary Lemmas concerning the classification rule are demonstrated in Section 4.2.

4.1 Covering number

Let us first recall that the ρ-*covering number* of a subset S of a metric space is defined as the smallest number of open balls of radius ρ whose union cover S. The logarithm of the ρ-covering number is generally called the metric entropy of S.

To bound the covering number of the manifold M, we shall need the following Lemma, which gives a lower bound on the volume of a geodesic ball in M, under the condition that the radius is small enough. The condition on the radius is linked with the geometry of the manifold, including its curvature. The curvature of a manifold measures the extent by which the parallel transport of a field along a small closed curve differs from the identity. There exists several equivalent notions to describe the curvature of a manifold, and in particular the *sectional curvature* which is used in the next Lemma. Given a point $p \in M$, consider a two-dimensional submanifold N_p of M consisting of geodesic arcs through p such that their tangent vectors at p form a two-dimensional subspace of the tangent space to M at p, i.e., a section π. Then, using on N_p the Riemannian metric induced by that of M, the sectional curvature of the section π is equal to the Gaussian curvature of N_p at p. A manifold is said of *constant curvature* if all its sectional curvatures are the same.

Lemma 4.1 *Let (M, g) be a compact Riemannian manifold without boundary of dimension d. Let δ be the infimum of the sectional curvatures of M. Let ρ be a strictly positive scalar such that*

$$\rho < \min\left\{\mathrm{inj}_g(M), \frac{\pi}{\sqrt{\delta}}, 2\pi\right\},$$

where $\mathrm{inj}_g(M)$ is the injectivity radius of M, and where we have set $\frac{\pi}{\sqrt{\delta}} = +\infty$ whenever $\delta \leq 0$. Then, for all $p \in M$, there exists a positive constant C independent of p such that

$$v_g\big(B_M(p, \rho)\big) \geq C\rho^d.$$

Proof: By the Günther–Bishop volume comparison Theorem (Chavel, 1993, Theo. 3.7), we have

$$v_g\big(B_M(p, \rho)\big) \geq V_\delta(\rho),$$

where $V_\delta(\rho)$ is the volume of the ball of radius ρ in the space of constant sectional curvature δ, i.e., the d-sphere of constant sectional curvature δ when $\delta > 0$; $\mathbb{R}^d$ when $\delta = 0$; and the hyperbolic space of constant sectional δ when $\delta < 0$.

Now we proceed to derive lower bounds on $V_\delta(\rho)$. To this aim, following Chavel (1993, p. 117), the volume $V_\delta(\rho)$ may be evaluated as follows:

$$V_\delta(\rho) = c_{d-1} \int_0^\rho S_\delta^{d-1}(t)\mathrm{d}t,$$

where

$$S_\delta(t) = \begin{cases} \frac{1}{\sqrt{\delta}} \sin(\sqrt{\delta}t), & \text{if } \delta > 0, \\ t, & \text{if } \delta = 0, \\ \frac{1}{\sqrt{-\delta}} \sinh(\sqrt{-\delta}t), & \text{if } \delta < 0, \end{cases}$$

and where c_{d-1} is the volume of the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$.

First of all, observe that, in the case where $\delta < 0$, we have $V_\delta(\rho) \geq V_0(\rho)$ since $\sinh(u) \geq u$ for all $u \geq 0$. Second, in the case where $\delta > 0$, we have $V_0(\rho) \geq V_\delta(\rho)$ since $\frac{1}{\sqrt{\delta}} \sin(\sqrt{\delta}t) \leq t$ for all $t \geq 0$. Consequently, it suffices to bound from below $V_\delta(\rho)$ in the case where $\delta > 0$.

To this aim, since $\rho < \frac{\pi}{\sqrt{\delta}}$, we have $\sqrt{\delta}t \leq \frac{\pi}{2}$ for all $t \leq \frac{\rho}{2}$. So using the inequality $\sin u \geq \frac{2}{\pi}u$ for all $0 \leq u \leq \frac{\pi}{2}$, we obtain

$$\begin{aligned} V_\delta(\rho) &\geq V_\delta(\rho/2) \\ &\geq c_{d-1}\left(\frac{1}{\sqrt{\delta}}\right)^{d-1}\int_0^{\rho/2}\left(\frac{2}{\pi}\sqrt{\delta}t\right)^{d-1}\mathrm{d}t, \end{aligned}$$

leading to the lower bound

$$V_\delta(\rho) \geq \frac{c_{d-1}}{d}\left(\frac{2}{\pi}\right)^{d-1}\left(\frac{\rho}{2}\right)^d, \tag{4.1}$$

which holds for all δ. □

Lemma 4.2 *Let (M, g) be a compact Riemannian manifold without boundary of dimension d. Let δ be the infimum of the sectional curvatures of M and let $\mathcal{N}(\rho)$ be the ρ-covering number of M. If ρ is such that*

$$0 < \rho < \min\left\{\mathrm{inj}_g(M), \frac{\pi}{\sqrt{\delta}}, 2\pi\right\},$$

where $\mathrm{inj}_g(M)$ is the injectivity radius of M, and where we have set $\frac{\pi}{\sqrt{\delta}} = +\infty$ whenever $\delta \leq 0$, then there exists a positive constant C such that

$$\mathcal{N}(\rho) \leq C\rho^{-d}.$$

Proof: Consider a maximal set of points $\{p_i;\ i \geq 1\}$ such that $d_g(p_i, p_j) > \rho$ for all $i \neq j$. Then $M \subset \cup_{i\geq 1} B_M(p_i, \rho)$ otherwise there would exist a point p on M such that p_i, $d_g(p, p_i) > \rho$ for all points p_i, which is not possible by the definition of the set $\{(p_i);\ i \geq 1\}$. Furthermore, since M is compact, there exists an integer N such that, after sorting the p_i's, we have

$$M \subset \cup_{i=1}^N B_M(p_i, \rho).$$

But $\cup_{i=1}^N B_M(p_i, \rho/2) \subset M$, and $B_M(p_i, \rho/2) \cap B_M(p_j, \rho/2) = \emptyset$ whenever $i \neq j$. As a consequence, we obtain that

$$\sum_{i=1}^N v_g\big(B_M(p_i, \rho/2)\big) \leq \mathrm{Vol}_g(M),$$

where $\mathrm{Vol}_g(M)$ is the volume of M. But from Lemma 4.1, there exists a positive constant C such that

$$v_g\big(B_M(p_i, \rho/2)\big) \geq C\rho^d,$$

for all $i = 1, \ldots, N$. Consequently we have

$$N \leq \frac{\mathrm{Vol}_g(M)}{C}\rho^{-d},$$

hence the Lemma. □

4.2 Auxiliary results

Consider the classification rule

$$g_n(p) = \begin{cases} 0, & \text{if } \frac{\sum_{i=1}^n \mathbf{1}_{\{Y_i=0\}} K_{h_n}(p,X_i)}{n\mathbb{E}K_{h_n}(p,X)} \geq \frac{\sum_{i=1}^n \mathbf{1}_{\{Y_i=1\}} K_{h_n}(p,X_i)}{n\mathbb{E}K_{h_n}(p,X)} \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, this classification rule is equivalent to g_n^0 defined in (3.1). Now we define the function η_n on M by

$$\eta_n(p) = \frac{\sum_{j=1}^n Y_j K_{h_n}(p, X_j)}{n\mathbb{E}K_{h_n}(p, X)},$$

and we shall denote by $\eta(p)$ the conditional probability that Y is 1 given $X = p$, i.e.,

$$\eta(p) = \mathbb{P}\{Y = 1|X = p\} = \mathbb{E}\left[Y|X = p\right].$$

According to Theorem 2.3 in Devroye et al. (1996, Chap. 2, p. 17), the Theorem will be proved if we show that

$$\int_M |\eta(p) - \eta_n(p)|\, \mu(\mathrm{d}p) \to 0 \quad \text{with probability one as } n \to \infty, \tag{4.2}$$

where μ is the probability measure of the random variable X.

Lemma 4.3 *Let $K_h(p, \cdot)$ be a kernel on M of the form given by (2.1). Let X be a random variable valued in M with probability measure μ. Then there exists a constant $C > 0$ depending only on K and on the geometry of M such that:*

$$\sup_{q\in M} \int_M \frac{K_h(p, q)}{\mathbb{E}K_h(p, X)} \mu(\mathrm{d}p) \leq C.$$

Proof: First of all, we have

$$\int_M \frac{K_h(p, q)}{\mathbb{E}K_h(p, X)} \mu(\mathrm{d}p) = \int_{B_M(q,h)} \frac{K_h(p, q)}{\mathbb{E}K_h(p, X)} \mu(\mathrm{d}p).$$

Next, cover the geodesic ball $B_M(q, h)$ by $\mathcal{N}_B$ geodesic balls centered at points p_i of $B_M(q, h)$ and of radius $\frac{h}{4}$. Then we start with the following inequality:

$$\begin{aligned} &\int_M \frac{K_h(p, q)}{\mathbb{E}K_h(p, X)} \mu(\mathrm{d}p) \\ &\quad \leq \sum_{i=1}^{\mathcal{N}_B} \int_{B_M(p_i,h/4)} \frac{K_h(p, q)}{\mathbb{E}K_h(p, X)} \mu(\mathrm{d}p) \\ &\quad = \sum_{i=1}^{\mathcal{N}_B} \int_{B_M(p_i,h/4)} \frac{\sup_{p\in B_M(p_i,h/4)} K_h(p, q)}{\mathbb{E}K_h(p, X)} \mu(\mathrm{d}p). \end{aligned} \tag{4.3}$$

Now we proceed to bound the two terms in the ratio under the integral above.

First of all, since $K_h(\cdot, q)$ is supported by $B_M(q, h)$, we have for all $i = 1, \ldots, \mathcal{N}_B$, and all $q \in M$:

$$\begin{aligned}
\sup_{p \in B_M\left(p_i, \frac{h}{4}\right)} K_h(p, q) &\le \sup_{p \in M} \sup_{q \in B_M(p,h)} K_h(p, q) \\
&\le \left(\sup_{p \in M} \sup_{q \in B_M(p,h)} \theta_p^{-1}(q) \right) \frac{1}{h^d} \sup_{\|x\| \le h} K\left(\frac{\|x\|}{h} \right) \\
&\le \left(\sup_{p \in M} \sup_{q \in B_M(p,h_0)} \theta_p^{-1}(q) \right) \frac{1}{h^d} \sup_{\|x\| \le 1} K\left(\|x\| \right) \\
&= C_1 \frac{1}{h^d},
\end{aligned} \tag{4.4}$$

where we have set

$$C_1 = \left(\sup_{p \in M} \sup_{q \in B_M(p,h_0)} \theta_p^{-1}(q) \right) \sup_{\|x\| \le 1} K\left(\|x\| \right),$$

and where h_0 is the constant defined by (2.2).

Second, for all $p \in M$, we have

$$\begin{aligned}
\mathbb{E} K_h(p, X) &= \int_M K_h(p, q) \mu(\mathrm{d}q) \\
&\ge \int_{B_M(p,h/2)} \theta_p^{-1}(q) \frac{1}{h^d} K\left(\frac{d_g(q, p)}{h} \right) \mu(\mathrm{d}q) \\
&\ge \left(\inf_{p \in M} \inf_{q \in B_M(p,h/2)} \theta_p^{-1}(q) \right) \frac{1}{h^d} \inf_{q \in B_M(p,h/2)} K\left(\frac{d_g(q, p)}{h} \right) \\
&\qquad \times \int_{B_M\left(p, \frac{h}{2}\right)} \mu(\mathrm{d}q) \\
&\ge \left(\inf_{p \in M} \inf_{q \in B_M(p,h_0)} \theta_p^{-1}(q) \right) \frac{1}{h^d} \inf_{\|x\| \le 1/2} K\left(\|x\| \right) \int_{B_M\left(p, \frac{h}{2}\right)} \mu(\mathrm{d}q) \\
&= C_2 \frac{1}{h^d} \mu\left(B_M\left(p, \frac{h}{2} \right) \right),
\end{aligned}$$

where

$$C_2 = \left(\inf_{p \in M} \inf_{q \in B_M(p,h_0)} \theta_p^{-1}(q) \right) \inf_{\|x\| \le 1/2} K\left(\|x\| \right).$$

Now, noting that for all $p \in B_M\left(p_i, \frac{h}{4}\right)$ we have $B_M\left(p_i, \frac{h}{4}\right) \subset B_M(p, \frac{h}{2})$, we obtain

$$\mathbb{E} K_h(p, X) \ge C_2 \frac{1}{h^d} \mu\left(B_M(p_i, h/4) \right), \tag{4.5}$$

for all $p \in B_M\left(p_i, \frac{h}{4}\right)$.

Reporting (4.2) and (4.5) yields

$$\begin{aligned}\int_M \frac{K_h(p,q)}{\mathbb{E}K_h(p,X)}\mu(\mathrm{d}p) &\leq \sum_{i=1}^{\mathcal{N}_B} \frac{C_1}{C_2} \int_{B_M(p_i,h/4)} \frac{\mu(\mathrm{d}p)}{\mu\left(B_M(p_i,h/4)\right)} \\ &= \frac{C_1}{C_2}\mathcal{N}_B\end{aligned}$$

for all $q \in M$. Now, applying Lemma 4.2 to $B_M(q,h)$, and since $\mathrm{Vol}_g(B_M(q,h)) = O(h^d)$, where the constant in $O(h^d)$ can be made uniform in q since M is closed, we obtain that there exists a constant C such that $\mathcal{N}_B \leq C$. Hence the Lemma. □

From now on, μ will denote the probability measure of X.

Lemma 4.4 *If* $h_n \to 0$ *then*

$$\int_M |\eta(p) - \mathbb{E}\eta_n(p)|\,\mu(\mathrm{d}p) \to 0$$

as $n \to \infty$.

Proof: Let $\varepsilon > 0$. Since M is compact, the set of continuous functions on M is dense in $L^1(M,\mu)$, and so there exists a continuous function r such that

$$\int_M |\eta(p) - r(p)|\mu(\mathrm{d}p) \leq \varepsilon.$$

First of all, we have

$$\begin{aligned}&\int_M |\eta(p) - \mathbb{E}\eta_n(p)|\mu(\mathrm{d}p) \\ &\quad\leq \int_M |\eta(p) - r(p)|\mu(\mathrm{d}p) + \int_M |r(p) - \mathbb{E}\eta_n(p)|\mu(\mathrm{d}p) \qquad (4.6)\\ &\quad\leq \varepsilon + \int_M |r(p) - \mathbb{E}\eta_n(p)|\mu(\mathrm{d}p).\end{aligned}$$

For the second term in the right-hand side of (4.6), we may write

$$\begin{aligned}&\int_M |r(p) - \mathbb{E}\eta_n(p)|\mu(\mathrm{d}p) \\ &\quad= \int_M \left| r(p) - \int_M \eta(q)\frac{K_{h_n}(p,q)}{\mathbb{E}K_{h_n}(p,X)}\right|\mu(\mathrm{d}q) \\ &\quad\leq \int_M\int_M |r(p) - \eta(q)|\frac{K_{h_n}(p,q)}{\mathbb{E}K_{h_n}(p,X)}\mu(\mathrm{d}p)\mu(\mathrm{d}q) \\ &\quad\leq \int_M\int_M |r(p) - r(q)|\frac{K_{h_n}(p,q)}{\mathbb{E}K_{h_n}(p,X)}\mu(\mathrm{d}p)\mu(\mathrm{d}q) \qquad (4.7)\\ &\qquad+ \int_M\int_M |r(q) - \eta(q)|\frac{K_{h_n}(p,q)}{\mathbb{E}K_{h_n}(p,X)}\mu(\mathrm{d}p)\mu(\mathrm{d}q).\end{aligned}$$

Now we proceed to prove that the two terms in the right-hand side of (4.7) are bounded from above by a constant multiple of ε for all n large enough.

Since the function r is continuous and since M is compact, r is uniformly continuous so there exists $\rho > 0$ such that $|r(q) - r(p)| < \varepsilon$ for all p and q in M with $d_g(p, q) < \rho$. Thus

$$\begin{aligned} &\int_M \int_M |r(p) - r(q)| \frac{K_{h_n}(p, q)}{\mathbb{E}K_{h_n}(p, X)} \mu(\mathrm{d}p)\mu(\mathrm{d}q) \\ &\quad \leq \int_M \int_{B_M(p,\rho)} |r(q) - r(p)| \frac{K_{h_n}(q, p)}{\mathbb{E}K_{h_n}(p, X)} \mu(\mathrm{d}q)\mu(\mathrm{d}p) \\ &\qquad + \int_M \int_{B^c_M(p,\rho)} |r(q) - r(p)| \frac{K_{h_n}(q, p)}{\mathbb{E}K_{h_n}(p, X)} \mu(\mathrm{d}q)\mu(\mathrm{d}p), \end{aligned} \tag{4.8}$$

where $B_M(p, \rho)$ and $B^c_M(p, \rho)$ denotes respectively the geodesic ball in M centered at p and of radius ρ, and its complement. But for n large enough, $h_n < \rho$ so $B_M(p, h_n) \subset B_M(p, \rho)$. Consequently, the second term in the right-hand side of (4.8) vanishes and we obtain

$$\begin{aligned} &\int_M \int_M |r(p) - r(q)| \frac{K_{h_n}(p, q)}{\mathbb{E}K_{h_n}(p, X)} \mu(\mathrm{d}p)\mu(\mathrm{d}q) \\ &\quad \leq \int_M \int_{B_M(p,\rho)} |r(q) - r(p)| \frac{K_{h_n}(q, p)}{\mathbb{E}K_{h_n}(p, X)} \mu(\mathrm{d}q)\mu(\mathrm{d}p) \\ &\quad \leq \varepsilon \int_M \int_{B_M(p,\rho)} \frac{K_{h_n}(q, p)}{\mathbb{E}K_{h_n}(p, X)} \mu(\mathrm{d}q)\mu(\mathrm{d}p) \\ &\quad = \varepsilon \int_M \int_{B_M(p,h_n)} \frac{K_{h_n}(q, p)}{\mathbb{E}K_{h_n}(p, X)} \mu(\mathrm{d}q)\mu(\mathrm{d}p) \\ &\quad = \varepsilon \operatorname{Vol}_g(M). \end{aligned} \tag{4.9}$$

Now for the second term in the right-hand side of (4.7), we have

$$\begin{aligned} &\int_M \int_M |r(q) - \eta(q)| \frac{K_{h_n}(p, q)}{\mathbb{E}K_{h_n}(p, X)} \mu(\mathrm{d}q)\mu(\mathrm{d}p), \\ &\quad \leq \sup_{q\in M} \int_M \frac{K_{h_n}(p, q)}{\mathbb{E}K_{h_n}(p, X)} \mu(\mathrm{d}p) \int_M |r(q) - \eta(q)| \mu(\mathrm{d}q) \\ &\quad \leq C\varepsilon \end{aligned} \tag{4.10}$$

for some constant C by Lemma 4.3.

Finally, reporting (4.10), (4.9), and (4.7) in (4.6) leads to the desired result. □

Lemma 4.5 *There exists a positive constant C such that*

$$\mathbb{E} \int_M |\eta_n(p) - \mathbb{E}\eta_n(p)| \, \mu(\mathrm{d}p) \leq C \left(\frac{1}{n} \mathcal{N}\left(\frac{h_n}{4} \right) \right)^{\frac{1}{2}}.$$

Proof: We have

$$
\begin{aligned}
\mathbb{E}\{|\eta_n(p)-\mathbb{E}\eta_n(p)|\} &\le \sqrt{\mathbb{E}\left\{|\eta_n(p)-\mathbb{E}\eta_n(p)|^2\right\}} \\
&= \left[\frac{\mathbb{E}\left\{\left(\sum_{j=1}^n Y_j K_{h_n}(p,X_j)-\mathbb{E}YK_{h_n}(p,X)\right)^2\right\}}{n^2\left(\mathbb{E}K_{h_n}(p,X)\right)^2}\right]^{1/2} \\
&= \left[\frac{\mathbb{E}\left\{\left(YK_{h_n}(p,X)-\mathbb{E}YK_{h_n}(p,X)\right)^2\right\}}{n\left(\mathbb{E}K_{h_n}(p,X)\right)^2}\right]^{1/2} \\
&\le \left[\frac{\mathbb{E}\left\{\left(YK_{h_n}(p,X)\right)^2\right\}}{n\left(\mathbb{E}K_{h_n}(p,X)\right)^2}\right]^{1/2} \\
&\le \left[\frac{\mathbb{E}K_{h_n}^2(p,X)}{n\left(\mathbb{E}K_{h_n}(p,X)\right)^2}\right]^{1/2}.
\end{aligned}
\tag{4.11}
$$

First of all, we have

$$
\begin{aligned}
\mathbb{E}K_{h_n}^2(p,X) &\le \sup_{q\in B_M(p,h_n)} K_{h_n}(p,q)\mathbb{E}K_{h_n}(p,X) \\
&\le \sup_{\|x\|\le 1} K(\|x\|)\left(\sup_{p\in M}\sup_{q\in B_M(p,h_0)}\theta_p^{-1}(q)\right)\frac{1}{h_n^d}\mathbb{E}K_{h_n}(p,X).
\end{aligned}
$$

Therefore

$$
\frac{\mathbb{E}K_{h_n}^2(p,X)}{n\left(\mathbb{E}K_{h_n}(p,X)\right)^2} \le \frac{C_1}{nh_n^d\mathbb{E}K_{h_n}(p,X)},
\tag{4.12}
$$

where $C_1 = \sup_{\|x\|\le 1} K(\|x\|)\left(\sup_{p\in M}\sup_{q\in B_M(p,h_0)}\theta_p^{-1}(q)\right)$.

Now we bound $\mathbb{E}K_{h_n}(p,X)$ as follows:

$$
\begin{aligned}
&\mathbb{E}K_{h_n}(p,X) \\
&\quad\ge \frac{1}{h_n^d}\int_{B_M\left(p,\frac{h_n}{2}\right)}\frac{1}{\theta_p(q)}K\left(\frac{d_g(q,p)}{h_n}\right)\mu(\mathrm{d}q) \\
&\quad\ge \frac{1}{h_n^d}\left(\inf_{p\in M}\inf_{q\in B_M(p,h_0)}\theta_p^{-1}(q)\right)\inf_{\|x\|\le 1/2}K(\|x\|)\mu\left(B_M\left(p,\frac{h_n}{2}\right)\right)
\end{aligned}
$$

and so

$$
\mathbb{E}K_{h_n}(p,X) \ge C_2\frac{1}{h_n^d}\mu\left(B_M\left(p,\frac{h_n}{2}\right)\right),
\tag{4.13}
$$

where $C_2 = \left(\inf_{p\in M}\inf_{q\in B_M(p,h_0)}\theta_p^{-1}(q)\right)\inf_{\|x\|\le 1/2}K(\|x\|)$.

From (4.11), (4.12) and (4.13), it follows that

$$\mathbb{E}\{|\eta_n(p) - \mathbb{E}\eta_n(p)|\} \le \frac{C_1}{C_2}\frac{1}{\sqrt{n}}\frac{1}{\sqrt{\mu\left(B_M\left(p, \frac{h_n}{2}\right)\right)}},$$

for all $p \in M$, and so

$$\int_M \mathbb{E}\{|\eta_n(p) - \mathbb{E}\eta_n(p)|\}\,\mu(\mathrm{d}p) \le \frac{C_1}{C_2}\sqrt{\mathrm{Vol}_g(M)}\frac{1}{\sqrt{n}}\left[\int_M \frac{\mu(\mathrm{d}p)}{\mu\left(B_M(p, h_n/2)\right)}\right]^{1/2},$$

by Cauchy–Schwarz. Now, using a cover of M by $\mathcal{N}\left(\frac{h_n}{4}\right)$ geodesic balls $B_M\left(p_i, \frac{h_n}{4}\right)$ centered at points p_i of M and of radius $\frac{h_n}{4}$, we obtain that

$$\begin{aligned}\int_M \frac{\mu(\mathrm{d}p)}{\mu\left(B_M(p, h_n/2)\right)} &\le \sum_{i=1}^{\mathcal{N}(h_n/4)} \int_{B_M(p_i, h_n/4)} \frac{\mu(\mathrm{d}p)}{\mu\left(B_M(p_i, h_n/4)\right)} \\ &= \mathcal{N}(h_n/4).\end{aligned}$$

Consequently

$$\int_M \mathbb{E}\{|\eta_n(p) - \mathbb{E}\eta_n(p)|\}\,\mu(\mathrm{d}p) \le \frac{C_1}{C_2}\sqrt{\mathrm{Vol}_g(M)}\left(\frac{1}{n}\mathcal{N}\left(\frac{h_n}{4}\right)\right)^{\frac{1}{2}}. \qquad \square$$

4.3 Proof of Theorem 3.1

We proceed to demonstrate (4.2), i.e., that

$$\int_M |\eta(p) - \eta_n(p)|\,\mu(\mathrm{d}p) \to 0 \quad \text{with probability one as } n \to \infty.$$

First of all, we have

$$\begin{aligned}&\mathbb{E}\int_M |\eta(p) - \eta_n(p)|\mu(\mathrm{d}p) \\ &\quad \le \int_M |\eta(p) - \mathbb{E}\eta_n(p)|\mu(\mathrm{d}p) + \mathbb{E}\int_M |\eta_n(p) - \mathbb{E}\eta_n(p)|\mu(\mathrm{d}p) \\ &\quad \le \int_M |\eta(p) - \mathbb{E}\eta_n(p)|\mu(\mathrm{d}p) + C_1\left(\frac{1}{n}\mathcal{N}\left(\frac{h_n}{4}\right)\right)^{\frac{1}{2}}\end{aligned}$$

for some positive constant C_1 by Lemma 4.5. Since $\mathcal{N}\left(\frac{h_n}{4}\right) = O\left(\frac{1}{h_n^d}\right)$ by Lemma 4.2, and since $nh_n^d \to \infty$ by assumption, it follows that

$$\frac{1}{n}\mathcal{N}\left(\frac{h_n}{4}\right) \to 0 \quad \text{as } n \to \infty.$$

Next, by applying Lemma 4.4, we obtain

$$\mathbb{E}\int_M |\eta(p) - \eta_n(p)|\mu(\mathrm{d}p) \to 0 \quad \text{as } n \to \infty.$$

Therefore, (4.2) will be proved if we show that

$$\int_M |\eta(p) - \eta_n(p)|\,\mu(\mathrm{d}p) - \mathbb{E}\int_M |\eta(p) - \eta_n(p)|\mu(\mathrm{d}p) \to 0$$

with probability one as $n \to \infty$. For this purpose, we shall use McDiarmid's inequality (McDiarmid, 1989) applied to the centered random variable

$$\int_M |\eta(p) - \eta_n(p)|\,\mu(\mathrm{d}p) - \mathbb{E}\int_M |\eta(p) - \eta_n(p)|\mu(\mathrm{d}p).$$

First of all, keep the data fixed at $(x_1, y_1), \ldots, (x_n, y_n)$ and replace the i$^{\text{th}}$ pair (x_i, y_i) by $(\bar{x}_i, \bar{y}_i)$, changing the value of $\eta_n(p)$ to $\bar{\eta}_i(p)$. Then we have

$$\begin{aligned}
\left|\int_M |\eta_n(p) - \eta(p)| d\mu(p) - \int_M |\bar{\eta}_i(p) - \eta(p)|\mu(\mathrm{d}p)\right| &\leq \int_M |\eta_n(p) - \bar{\eta}_i(p)|\mu(\mathrm{d}p) \\
&\leq \frac{2}{n}\sup_{q\in M}\int_M \frac{K_{h_n}(p,q)}{\mathbb{E}K_{h_n}(p,X)}\mu(\mathrm{d}p) \\
&\leq \frac{2C_1}{n}
\end{aligned}$$

using Lemma 4.3, for some positive constant C_1. So, applying McDiarmid's inequality (McDiarmid, 1989) yields

$$\begin{aligned}
&\mathbb{P}\left\{\int_M |\eta_n(p) - \eta(p)|\mu(\mathrm{d}p) \geq \varepsilon\right\} \\
&\quad \leq \mathbb{P}\left\{\int_M |\eta_n(p) - \eta(p)|\mu(\mathrm{d}p) - \mathbb{E}\int_M |\eta_n(p) - \eta(p)|\mu(\mathrm{d}p) \geq \frac{\varepsilon}{2}\right\} \\
&\quad \leq C\exp\left(-\varepsilon^2 n\right).
\end{aligned}$$

for all $\varepsilon > 0$. Finally, and using the Borel–Cantelli Lemma, we conclude that

$$\int_M |\eta(p) - \eta_n(p)|\,\mu(\mathrm{d}p) - \mathbb{E}\int_M |\eta(p) - \eta_n(p)|\mu(\mathrm{d}p) \to 0$$

with probability one as $n \to \infty$, which proves (4.2), and so the Theorem. □

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